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Relativistic two-time localization

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Abstract. We have earlier defined the quantum two-time localization problem as the minimizing of a particle's position spread about specified points at two distinct times. In the present article optimum localization is found for relativistic massive free particles. For short time intervals, spreading necessarily occurs at the speed of light while for long times the previously found diffusion-like behaviour is recovered. In defining relativistic localization, use is made of the work of Newton and Wigner; in particular, their restriction to the positive energy hyperboloid is found to be necessary to recover the non-relativistic limit of wavepacket spreading.

1. Introduction

How well can the position of a particle be specified at two separate times? In a previous publication [1] we gave precise meaning to this question and provided an answer for certain non-relativistic quantum systems. In particular, the specification criterion was based on the minimization of the sum of the (quadratic) position spreads about specified positions. In the present article the relativistic version of this problem will be posed and, for free massive particles, solved.

The results of this article confirm and amplify the picture of the relativistic to non-relativistic limit that was developed in [2] and [3]. There, modulo an analytic continuation, a particle is seen as instantaneously moving at the speed of light, reversing its direction at random *Poisson distributed* times with a rate proportional to the particle mass. The correlation length (or time) is the Compton wavelength ($1/m$) and, on scales long compared to this, the Poisson process goes over to a *Wiener process.* This limit holds for both the one (space) dimensional case, where the random walk is just that, and for the three space dimensional case where relativistically the 'walk' steps between alternate modes of wave propagation, although the non-relativistic limit allows the more concrete coordinate space walk interpretation. The diffusion coefficient for the Wiener process that results from this limit is the same as that found in the path integral and other characterizations of quantum stochastics, namely $\hbar/m$. The subtlety in our present work is the difficulty associated with relativistic definitions of position, and in our handling of this issue we follow Newton and Wigner [4]. Our conclusions, consistent with the picture just described, are that for short times you cannot prevent wavepackets from spreading at the speed of light while for longer times we recover our earlier result [1] that $\Delta x^2 \sim T$, rather like diffusion. A feature that plays an important role in our arguments is the restriction by Newton and Wigner to the positive energy hyperboloid; this is essential to making sense of the non-relativistic limit of Dirac-particle wavepacket expansion. Without that restriction the packet would spread at the speed of light

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indefinitely. We mention this because for some applications one uses negative *frequency* components in wave equation solutions, a use that can obscure the need for positive energies in what Newton and Wigner call 'elementary systems'. We will comment further on this point in our conclusions.

Aside from its intrinsic interest, the answer to the localization question forms one of the links in the chain of reasoning justifying the need for 'special' states in loosely time symmetric cosmologies, which is in turn relevant to our proposed measurement theory [5]. As such, the present paper is a step in that programme.

In the next section we formulate the localization criterion and review our non-relativistic results. In section 3 we solve the two-time localization problem for the relativistic scalar free particle and in section 4 deal with Dirac particles. Section 5 is a discussion of the results.

2. Formulation and non-relativistic result

We are given a particular Hamiltonian H and two events (x_j, t_j), $j=a, b$. The square spread about x_j (at time t_j) is defined to be

$$(\Delta x_j)^2 \equiv \int \mathrm{d}x |\psi_{t_j}(x)|^2 (x-x_j)^2 \qquad j=a, b \tag{1}$$

where $\psi_{t_j} = \exp(-\mathrm{i}t_j H/\hbar)\psi_0$ and $t=0$ is a fixed, fiducial time. Our goal is to minimize the sum $(\Delta x_a)^2 + (\Delta x_b)^2$. This is accomplished by varying ψ_0. Optimum localization is thus defined as the minimizing of the functional

$$W(\psi_0; x_a, t_a, x_b, t_b) \equiv (\Delta x_a)^2 + (\Delta x_b)^2. \tag{2}$$

By bringing the evolution operator $\exp(-\mathrm{i}tH/\hbar)$ to bear on x instead of on ψ_0, W can be written

$$W(\psi_0; x_a, t_a, x_b, t_b) = \langle \psi_0 | \hat{W} | \psi_0 \rangle \tag{3}$$

with

$$\hat{W} \equiv (\hat{x}(t_a) - x_a)^2 + (\hat{x}(t_b) - x_b)^2 \tag{4}$$

and $\hat{x}(t)$ is the time-dependent Heisenberg operator

$$\hat{x}(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar} \hat{x}\, \mathrm{e}^{-\mathrm{i}Ht/\hbar}. \tag{5}$$

The two-time localization problem is thus the finding of the ground state of $\hat{W}$. Of greatest immediate interest for us will be the value of W for that ψ_0, i.e. the lowest eigenvalue of $\hat{W}$, W_0. Its dependence on $T \equiv t_b - t_a$ will be of paramount importance.

In [1], the Hamiltonian $(1/2m)p^2 + (1/2)m\omega^2 x^2$ was studied and we found that

$$W_0 = \frac{\hbar}{m} \frac{|\sin \omega T|}{\omega} \tag{6}$$

so that

$$W_0 = \frac{\hbar T}{m} \qquad \text{free particle.} \tag{7}$$

An extension to WKB approximations for slowly varying potentials was made in [6].

3. Relativistic, spin-zero, two-time localization

Before one can localize at *two* times, one must give meaning to this concept at a single time. For this purpose we replace our non-relativistic position variable 'x' by that of Newton and Wigner [4]. There are two essential points that go into their formulation and we state them in terms of momentum, $p \in \mathcal{R}^3$, and momentum space wavefunctions $\phi(p)$. First, relativistic covariance fixes the momentum space volume element to be $\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3/p_0$, with

$$p_0 \equiv +(m^2+p_1^2+p_2^2+p_3^2)^{1/2} \tag{8}$$

rather than merely $\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3$, as would be the case non-relativistically. Secondly, the wavefunction is defined on the positive energy hyperboloid *only*. For the free particle, time evolution of $\phi(p)$ is given by $\phi(p)\,\mathrm{e}^{-\mathrm{i}tp_0}$, with $p_0>0$, as in (8). (Note that we will henceforth mostly take $\hbar=1$ and $c=1$.)

The approach of Newton and Wigner is first to postulate properties of 'localized states,' then to deduce the form those states must take as a result of these postulates and finally to define position operators as the operators having these states as eigenfunctions with appropriate eigenvalues. For spin-zero particles their result for the position operator q is

$$q\phi(p)=\left(\mathrm{i}\nabla_p-\mathrm{i}\frac{p}{2p_0^2}\right)\phi(p). \tag{9}$$

A word of notational caution is in order here. For us, q, $p \in \mathcal{R}^3$, while in Newton and Wigner contravariant and covariant vectors are distinguished and care must be exercised in conforming to their conventions in this regard. Using the definition of position space wavefunction,

$$\Phi(x,t)=(2\pi)^{-3/2}\int \phi(p)\exp(-\mathrm{i}tp_0+\mathrm{i}x\cdot p)\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3/p_0 \tag{10}$$

(where $x\in\mathcal{R}^3$ and $x\cdot p=x_1p_1+x_2p_2+x_3p_3$). Newton and Wigner show that the operator q takes the form

$$q\Phi(x)=x\Phi(x)+\frac{1}{8\pi}\int\frac{\exp(-m|x-y|)}{|x-y|}\nabla_y\Phi(y)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3. \tag{11}$$

The actual form of q, as given in (9) or (11), is determined by the factor p_0. For $|p|\ll m$, p_0 is essentially constant so that it is quite reasonable that the principal effect of the considerations of Newton and Wigner is to make q differ from multiplication by x only in a region on the order of the Compton wavelength $1/m$, as can be seen directly in (11).

We will formulate the relativistic two-time localization problem as we did in the non-relativistic case, except that the position operator in $\hat{W}$ (equation (4)) will be replaced by its relativistic counterpart. Our goal is therefore to find the lowest eigenvalue of the operator

$$\hat{W}=(\hat{q}(t_b)-x_b)^2+(\hat{q}(t_a)-x_a)^2 \tag{12}$$

with $\hat{q}(t)$ the appropriate Heisenberg operator.

For the non-relativistic case, the positions x_a and x_b did not affect W_0, as could have been anticipated (for the free particle) on the basis of Galilean invariance. By the same token, for timelike separated events we can go to the frame in which $x_a=x_b=0$.

Unlike Galilean relativity, transforming away from this frame does have consequences; however, our primary interest does not lie in this kinematical effect and we therefore take $x_a = x_b = 0$. Finally, for a time translation invariant system, there is no loss in generality in taking $t_b = T/2$ and $t_a = -T/2$. The operator to be studied is therefore

$$\hat{W} = \hat{q}(T/2)^2 + \hat{q}(-T/2)^2. \tag{13}$$

In momentum space the time evolution of $\hat{q}$ is easy to deduce. In that representation we have

$$\begin{aligned} q(t) &= e^{ip_0 t}\left(i\nabla_p - \frac{ip}{2p_0^2}\right)e^{-ip_0 t} \\ &= -\frac{ip}{2p_0^2} + e^{ip_0 t} i\nabla_p (e^{-ip_0 t}) \\ &= q + \frac{pt}{p_0}. \end{aligned} \tag{14}$$

(The caret will be dropped henceforth.) For W this gives

$$W = 4\left[\frac{1}{2}q^2 + \frac{1}{2}\left(\frac{T}{2}\right)^2 \frac{p^2}{p^2+m^2}\right]. \tag{15}$$

From (9) it follows that p and q have the usual commutation relations. Moreover, the spectrum of q is the same as that of x, namely the entire real line. For convenience in analysing W we will rename q, π and p, y. The resulting investigation of the ground state of W is a straightforward problem in quantum mechanics with

$$\frac{W}{4} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{T}{2}\right)^2 \frac{y^2}{y^2+m^2} \tag{16}$$

and π and y canonically conjugate. The cases of interest are large and small T. As will be seen self-consistently, for large T, y is small. We examine therefore

$$H_{\text{Large } T} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{T}{2m}\right)^2 y^2.$$

The ground state of this operator has eigenvalue $\frac{1}{2}(T/2m)$. The relevant range of y is $1/\sqrt{(T/2m)}$, showing that our self-consistency condition is $mT \gg 1$. (Equivalently: $mc^2T \gg \hbar$.) From the ground state energy of $H_{\text{Large } T}$ we get

$$W_0 = 3T/m \qquad \text{large } T \tag{17}$$

in agreement with (7) (when $\hbar$ is restored and allowance made for dimension).

For small T the significant range of y is large compared to m so that W essentially describes a free particle (thinking of π as momentum) in a constant potential $T^2/8$. The lowest energy of such a particle is simply $T^2/8$ and its wavefunction is spread everywhere (in 'y'). It follows that

$$W_0 = T^2/2 \qquad \text{small } T. \tag{18}$$

This result is new and involves relativistic effects in an essential way. Large 'y' means large momentum and the solution (18) indicates that wavepacket spreading occurs at the speed of light. ($\Delta x^2 \sim T^2$ with the factor $\frac{1}{2}$ arising from squeezing at $t=0$, the halfway time.)

The qualitative validity of our conclusions about the operator W in (15) can be checked by comparing W to the exactly solvable $\frac{1}{2}\pi^2 - v_0/(\cosh ay)^2$ for appropriate parameter values. Another check is to use the Rayleigh–Ritz variational principle, taking as test function $\sqrt{\alpha/\pi}\exp(-\alpha y^2/2)$, and varying with respect to alpha. We obtain the rigorous bound

$$E_0 \equiv \frac{W_0}{4} \leqslant \frac{\alpha}{4m^2} + \tfrac{1}{8}T^2 - \tfrac{1}{8}T^2\sqrt{\alpha\pi}\,\mathrm{e}^{\alpha}\,\mathrm{erfc}(\sqrt{\alpha}). \tag{19}$$

For small and large T, good values of α can be found analytically (using the asymptotics of the error function). The corresponding upper bounds are as follows

$$E_0 < \tfrac{1}{2}T^2 \qquad E_0 < \frac{T}{4m} - \frac{3}{8m^2} + \frac{15}{8Tm^3}.$$

In figure 1 the dashed line shows the minimum of the above two expressions. The solid line is a numerically determined minimum. As can be seen, the large and small T asymptotics are captured in equations (17) and (18).

A stochastic process-path integral picture of the transition from (19) to (18) will be deferred to the final section where previous work [2, 3] provides background to such discussion.

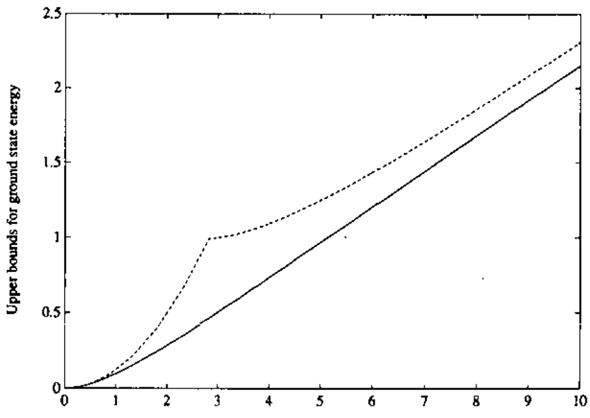

Figure 1. Upper bounds for the ground state of the operator $W/4$ given in equation (16). The solid line is based on numerical work using (19) and the dashed line on the analytic forms given in the text.

4. Dirac equation two-time localization

The Dirac equation can be written

$$\mathrm{i}\frac{\partial\psi}{\partial t} = m\sigma_x\psi - \mathrm{i}Q\sigma_z\psi \tag{20}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \partial = \partial/\partial x \tag{21}$$

$$Q = \partial - \mathrm{i}eA \qquad \text{one space dimension} \tag{22}$$

$$Q = \boldsymbol{\sigma}\cdot(\partial - \mathrm{i}eA) \qquad \text{three space dimensions} \tag{23}$$

ψ_i $(i=1,2)$ are complex valued functions in (dimension) $d=1$ and complex valued (2-component) spinors in $d=3$. Note that the σs in (23) and the σs in (20) commute since they act on different vector spaces. The propagator is a matrix that provides time evolution through

$$\psi(x,t) = \int \mathrm{d}y\, K(x,t;y)\psi(y,0). \tag{24}$$

We will only present our calculations for the one space dimension equation. For three space dimensions essentially the same manipulations are performed with slightly more elaborate objects. This parallel structure is a virtue of the Weyl representation (equations (20)–(24)).

In our previous section, although we quoted the full form of the Newton–Wigner position operator, in the end that specific form played no role. This is because the additional term ($p/2p_0^2$ of equation (9)) was time independent and only $\mathrm{i}\nabla_p$ was important. For a particle with non-zero spin the Newton–Wigner position is again basically $\mathrm{i}\nabla_p$ plus time-independent pieces. For this reason we will take $\mathrm{i}\nabla_p$, or the usual 'x', as position. Nevertheless, there is still an important lesson in [4] for the present case: energy should be positive. The propagator given in (24) acts on both positive and negative energies and without the positive energy restriction we would not recover the non-relativistic limit, as will be demonstrated shortly.

As usual we want the lowest eigenvalue W_0 of the operator

$$W = x(T/2)^2 + x(-T/2)^2. \tag{25}$$

Proceeding blindly, the equation of motion of the Heisenberg operator x is

$$\dot{x} = \mathrm{i}[H,x] = \sigma_z \tag{26}$$

so that

$$x(t) = x + \sigma_z t. \tag{27}$$

This would yield

$$W = 2\left(x^2 + \left(\frac{T}{2}\right)^2\right) \tag{28}$$

whose lowest eigenvalue is simply $T^2/2$, for all T. This is not the correct non-relativistic limit.

The positive energy restriction can be implemented in various ways. We will do so by projecting out of the propagator K the relevant portion and then using this to modify σ_z (of equations (26) and (27)) in the appropriate way.

It is useful to write the equation for K in the form

$$\frac{\partial K}{\partial t} = -\mathrm{i}LK \tag{29}$$

with

$$L \equiv m\sigma_x + p\sigma_z. \tag{30}$$

By applying $\partial/\partial t$ to (29), it follows that

$$K = K_1\, e^{-i\lambda t} + K_2\, e^{i\lambda t} \tag{31}$$

with

$$\lambda = +\sqrt{L^2} = +\sqrt{m^2 - \partial^2/\partial x^2}. \tag{32}$$

The operator λ (which is essentially p_0) is positive and the breakup in (31) is thus seen to be the breakup into positive and negative energy components. Incidentally, as shown in [3], the usual non-relativistic propagator is obtained by applying the phase factor e^{+imt}, since

$$\lambda - m \sim -\frac{1}{2m}\frac{\partial^2}{\partial x^2}. \tag{33}$$

(It is also of interest to note that the positive and negative transfer matrix eigenvalues of [2] correspond to positive and negative energies.)

The constant operators K_1 and K_2 of (31) are obtained from the boundary condition $K(0) = 1$ and from (29). These imply

$$K_1 = \frac{1}{2}\left(1 + \frac{L}{\lambda}\right). \tag{34}$$

K_1 has the property $K_1^2 = K_1$ and is therefore the projection onto the positive energy portion of the wavefunction.

The operator of interest is thus not the naively calculated x of (27), but $K_1 x K_1$. We consider first the action of K_1 on the velocity part of $x(t)$ and examine $K_1\sigma_z K_1$. A short calculation shows that

$$K_1\sigma_z K_1 = \frac{p}{\lambda} K_1. \tag{35}$$

As for the Dirac particle in three dimensions, the one-dimensional particle moves only at $v = \pm c$, as follows from the spectrum of σ_z. However, we see from (35) that the positive energy requirement suppresses this additional *zitterbewegung* effect and gives us the same expression for velocity that we obtained in the scalar case, equation (14).

From here the analysis proceeds as in the scalar case. The projection by K_1 does affect the position part of $x(t)$ since x and K_1 (which is a function of $\partial/\partial x$) do not commute. However, when one is entirely confined to the range of K_1 the commutation relations of p and $K_1 x K_1$ are unchanged. (Consider $[K_1 x K_1, p] = K_1[x, p]K_1$.) It follows that for short times $W_0 \sim T^2$ and for long times $W_0 \sim T/m$.

5. Discussion

The result

$$W_0 = [(\Delta x_a)^2 + (\Delta x_b)^2]_{\min} \sim \begin{cases} T^2 & \text{short time} \\ T/m & \text{long time} \end{cases} \tag{36}$$

is consistent with the picture developed in [2] and [3]. In those references the following description of Dirac particle time evolution was given: the particle moves at the speed of light and reverses direction randomly with a rate m. This picture emerges from Feynman's Dirac equation path integral [7] and modulo an analytic continuation the reversals are Poisson distributed. The correlation length (or time) is $1/m$ so that for times short compared to this all motion is effectively at the speed of light. When the process is smeared on timescales long compared to $1/m$ one gets Wiener-process-like behaviour which is consistent with the diffusion-like result of the second case of equation (36).

One can also look on the result (36) as the placing of a limit on the otherwise infinite velocity of spreading that would be predicted by $\Delta x^2 \sim T/m$ if one would consider this in the limit $T \to 0$; the relation of this to the imposition of a cutoff on the otherwise infinite velocities predicted by continuum Brownian motion theories is discussed in [2] and [3].

A point that emerges in the course of our Dirac-particle calculation is the necessity of the positive energy hyperboloid restriction, as given by Newton and Wigner. Without this, our results on wavepacket spreading would not have had the correct non-relativistic limit. This is of interest because in some contexts one makes use of negative frequencies. However, like Newton and Wigner, the object that we study is an 'elementary system', effectively an object on which the Poincaré group acts irreducibly, and as such one for which the physical requirement of positive *energy* obtains. (How such an object would be affected by the apparatus that would measure the abstractly defined position could conceivably be part of a larger dynamical theory [8], but is not considered in this article.) We remark in passing that for spin-$\frac{1}{2}$ the Newton–Wigner operators are the same as the Foldy–Wouthuysen 'mean position' operators [8].

Finally we comment on the relevance of this work to the programme [5]. The use made there of the two-time boundary problem involves the compulsory growth of W_0 with T for free particles, as compared with the asymptotic constancy of this quantity for the bound state situation. The present paper shows that relativistic considerations are pretty much irrelevant for the large time two-time localization problem and as such the arguments of [5] in this regard remain intact.

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